

AN ESTIMATE ON THE NODAL SET OF EIGENSPINORS ON CLOSED SURFACES

VOLKER BRANDING

ABSTRACT. We use a modified Bochner technique to derive an inequality relating the nodal set of eigenspinors to eigenvalues of the Dirac operator on closed surfaces. In addition, we apply this technique to solutions of similar spinorial equations.

1. INTRODUCTION AND RESULTS

Throughout this note we assume that (M, g) is a closed, oriented surface with a Riemannian metric g . The bundle $SO(M)$ of oriented orthonormal frames is an S^1 -principal bundle over the surface M . Let $\Theta : S^1 \rightarrow S^1$ be the nontrivial double covering of S^1 . A spin structure on M is an S^1 -principal bundle $Spin(M)$ over M together with a twofold covering map $\theta : Spin(M) \rightarrow SO(M)$ such that the diagram

$$\begin{array}{ccc} Spin(M) \times S^1 & \longrightarrow & Spin(M) \\ \downarrow \theta \times \Theta & & \downarrow \theta \\ SO(M) \times S^1 & \longrightarrow & SO(M) \end{array} \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \quad M$$

commutes. Every orientable surface admits a spin structure, the number of possible spin structure is equal to the number of elements in $H^1(M, \mathbb{Z}_2)$. On the spinor bundle ΣM we have a metric connection ∇ and a hermitian scalar product. Sections in the spinor bundle are called *spinors*. Moreover, there exists the canonical splitting of the spinor bundle ΣM into the *bundle of positive spinors* $\Sigma^+ M$ and the *bundle of negative spinors* $\Sigma^- M$, that is $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$. In addition, we have the Clifford multiplication of spinors with tangent vectors, denoted by $X \cdot \psi$ for $X \in TM$ and $\psi \in \Gamma(\Sigma M)$. Clifford Multiplication is skew-symmetric

$$\langle X \cdot \psi, \xi \rangle = -\langle \psi, X \cdot \xi \rangle$$

and satisfies the Clifford relations

$$X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2g(X, Y)\psi$$

for $X, Y \in TM$ and $\psi, \xi \in \Sigma M$.

Definition 1.1. The *Dirac operator* D maps smooth sections of ΣM to smooth sections of ΣM and is given by

$$D\psi := e_1 \cdot \nabla_{e_1} \psi + e_2 \cdot \nabla_{e_2} \psi,$$

where e_1, e_2 is a local orthonormal frame of TM .

The Dirac operator is a first order, elliptic operator, which is self-adjoint with respect to the L^2 -norm. Thus, from general spectral theory we know that the spectrum of the Dirac-operator is real, discrete and tends rapidly to infinity. In addition, it is well known that the Dirac-operator has its spectrum on the whole real line.

Date: October 6, 2015.

2010 Mathematics Subject Classification. 53C27, 58J05, 58C40.

Key words and phrases. Dirac operator; closed surface; eigenspinor; nodal set.

The square of the Dirac operator satisfies the so-called *Schrödinger-Lichnerowicz* formula

$$D^2 = \nabla^* \nabla + \frac{R}{4}, \quad (1.1)$$

where R denotes the scalar curvature of the manifold.

For more background material on spin geometry and the Dirac operator we refer the reader to the books [19] and [12].

Definition 1.2. A spinor $\psi \in \Gamma(\Sigma M)$ is called eigenspinor with eigenvalue λ if it satisfies

$$D\psi = \lambda\psi. \quad (1.2)$$

In particular, an eigenspinor corresponding to the eigenvalue $\lambda = 0$ is called *harmonic*.

In general, the spectrum of the Dirac operator D cannot be computed explicitly. There are only few manifolds with high symmetry, who allow to explicitly determine the spectrum, for example flat tori [11] and round spheres [4].

However, it is possible to estimate the spectrum. A fundamental inequality for the eigenvalues of the Dirac operator is the so-called Friedrich's inequality [10]

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M R, \quad (1.3)$$

where n is the dimension of the manifold M .

On closed surfaces, the following inequality was given by Bär in [3]

$$\lambda^2 \geq \frac{2\pi\chi(M)}{\text{Vol}(M, g)}, \quad (1.4)$$

where $\chi(M)$ is the Euler characteristic of M . It directly follows from (1.4) that there do not exist harmonic spinors on S^2 . However, on surfaces of genus $g \geq 1$ there always exist a metric and a spin structure admitting harmonic spinors, see [17], Proposition 2.4, [14], Theorem 6.2.1 and [7]. It is also possible to give estimates on the first non-zero eigenvalue of the Dirac operator that include the spin structure, see for example [2].

For more details on the spectrum of the Dirac operator see the book [14].

Basically, inequality (1.4) gives information about small eigenvalues of the Dirac operator. In this note we will derive an inequality that provides properties of large eigenvalues. On closed surfaces the nodal set of eigenspinors is discrete [6], which enables us to prove the following

Theorem 1.3. *Suppose that (M, g) is a closed spin surface with fixed spin structure. Then the k -th eigenvalue λ_k of the Dirac operator D satisfies the following inequality*

$$\lambda_k^2 \geq \frac{2\pi\chi(M)}{\text{Vol}(M, g)} + \frac{4\pi N_k}{\text{Vol}(M, g)}, \quad (1.5)$$

where $\chi(M)$ is the Euler characteristic of M . Moreover, N_k denotes the sum of the order of the zero's of an eigenspinor ψ_k belonging to the k -th eigenvalue of the Dirac operator, that is

$$N_k = \sum_{p \in M, |\psi_k|(p)=0} n_p. \quad (1.6)$$

Corollary 1.4. *Of course, (1.5) can also be interpreted as an inequality for the nodal set of an eigenspinor ψ_k belonging to the k -th eigenvalue of the Dirac operator*

$$N_k \leq \frac{\text{Vol}(M, g)\lambda_k^2}{4\pi} - \frac{\chi(M)}{2}. \quad (1.7)$$

We will discuss a similar inequality for harmonic spinors, twistor spinors and solutions of a semi-elliptic Dirac equation.

Remark 1.5. Note that (1.7) also holds if λ_k is an eigenvalue with higher multiplicity. In this case we could consider a linear combination of eigenspinors and the nodal set could depend on the particular linear combination. However, the estimate (1.7) holds for all linear combinations.

Remark 1.6. We cannot expect that a similar inequality holds in higher dimensions since the nodal set of eigenspinors is no longer discrete [6].

Acknowledgements: The author would like to thank Andreas Hermann for several useful comments on a first draft of this note.

2. PROOF OF THE MAIN THEOREM

By the main result of [6] we know that on a two-dimensional manifold the zero-set of eigenspinors is discrete. In the following we will make use of the energy-momentum tensor $T(X, Y)$

$$T(X, Y) := \langle X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \psi \rangle. \quad (2.1)$$

This tensor arises if one varies the functional $E(\psi) = \int_M \langle \psi, D\psi \rangle d\mu$ with respect to the metric. The following Lemma can be found in [13], Lemma 5.1, see also [18].

Lemma 2.1. *For all $\psi \in \Gamma(\Sigma M)$ the following inequality holds*

$$\frac{\langle \psi, D^2 \psi \rangle}{|\psi|^2} \geq \frac{R}{4} + \frac{|T|^2}{4|\psi|^4} - \Delta \log |\psi|. \quad (2.2)$$

Proof. We define a modified connection on ΣM by

$$\tilde{\nabla}_X \psi := \nabla_X \psi - 2\alpha(X)\psi - \beta(X) \cdot \psi - X \cdot \alpha \cdot \psi,$$

with a one-form α and a symmetric $(1, 1)$ -tensor β given by

$$\alpha := \frac{d|\psi|^2}{2|\psi|^2}, \quad \beta := -\frac{T(\cdot, \cdot)}{2|\psi|^2}.$$

By a direct computation we then find summing over repeated indices

$$|\tilde{\nabla} \psi|^2 = |\nabla \psi|^2 + 2|\alpha|^2 |\psi|^2 + |\beta|^2 |\psi|^2 - 4\alpha_{e_i} \langle \nabla_{e_i} \psi, \psi \rangle + 2\langle \beta(e_i) \cdot \nabla_{e_i} \psi, \psi \rangle.$$

Moreover, we have

$$\begin{aligned} |\nabla \psi|^2 &= \langle \psi, D^2 \psi \rangle - \frac{R}{4} |\psi|^2 + \frac{1}{2} \Delta |\psi|^2, \\ \alpha_{e_i} \langle \nabla_{e_i} \psi, \psi \rangle &= |\alpha|^2 |\psi|^2 = \frac{|d|\psi|^2|^2}{4|\psi|^2}, \\ \langle \beta(e_i) \cdot \nabla_{e_i} \psi, \psi \rangle &= -\frac{|T|^2}{4|\psi|^2}. \end{aligned}$$

Thus, we arrive at

$$0 \leq |\tilde{\nabla} \psi|^2 = \langle \psi, D^2 \psi \rangle - \frac{R}{4} |\psi|^2 + \frac{1}{2} \Delta |\psi|^2 - \frac{|d|\psi|^2|^2}{2|\psi|^2} - \frac{|T|^2}{4|\psi|^2}$$

yielding the result. \square

The following Lemma will be the key-tool for the further analysis. For the sake of completeness we also present a proof, where we follow [21].

Lemma 2.2. *Suppose M is a closed Riemannian surface. If the zero set of $|\psi|$ is discrete and $|\psi|$ does not vanish identically, then the following equality holds*

$$\int_M \Delta \log |\psi| = -2\pi \sum_{p \in M, |\psi|(p)=0} n_p, \quad (2.3)$$

where n_p is the order of $|\psi|$ at the point p .

Proof. Since the zeros of $|\psi|$ are isolated and M is compact, the number of zeros p_1, \dots, p_k is finite in M . Let $D_\varepsilon(p_j)$ be a small disc of radius $\varepsilon > 0$ around p_j . Applying the divergence theorem we get

$$\int_M \Delta \log |\psi| dM = \lim_{\varepsilon \rightarrow 0} \sum_{j=1}^k \int_{D_\varepsilon(p_j)} \Delta \log |\psi| dM = - \lim_{\varepsilon \rightarrow 0} \sum_{j=1}^k \int_{\partial D_\varepsilon(p_j)} \frac{\partial}{\partial r} \log |\psi| r d\theta,$$

where $\frac{\partial}{\partial r}$ denotes the radial derivative. Using a local Taylor expansion for each j we get

$$\int_{\partial D_\varepsilon(p_j)} \frac{\partial}{\partial r} \log |\psi| r d\theta = 2\pi n_{p_j} + O(\varepsilon),$$

where n_j denotes the order of the first non-vanishing term in the Taylor expansion of ψ in p_j . By letting $\varepsilon \rightarrow 0$ we thus obtain

$$\int_M \log |\psi| dM = -2\pi \sum_{j=1}^k n_{p_j} = -2\pi \sum_{p \in M, |\psi|(p)=0} n_p,$$

which proves the Lemma. \square

Finally, we apply Lemma 2.1 in the case that ψ is an eigenspinor. We can estimate the energy momentum tensor by $|T|^2 \geq 2\lambda^2 |\psi|^4$. Thus, from (2.2) we obtain

$$\lambda^2 \geq K - 2\Delta \log |\psi|, \quad (2.4)$$

where K denotes the Gaussian curvature of M . Integrating over the surface M and using (2.3) completes the proof of Theorem 1.3.

3. NODAL SETS OF SOLUTIONS OF SIMILAR EQUATIONS

In this section we are concerned with the nodal set of *harmonic spinors*, *twistor spinors*, *Killing spinors* and solutions of a non-linear Dirac equation. Most of the results presented in this section are well-known in the literature.

Lemma 3.1 (Bochner formula). *For an arbitrary spinor $\psi \in \Gamma(\Sigma M)$ the following Bochner formula holds*

$$\Delta \log |\psi| = \frac{K}{2} - \frac{\langle \psi, D^2 \psi \rangle}{|\psi|^2} + \frac{|\nabla \psi|^2}{|\psi|^2} - \frac{1}{2} \frac{|d|\psi|^2|^2}{|\psi|^4}, \quad (3.1)$$

where K is the Gaussian curvature of the surface.

Proof. This follows by a direct calculation using (1.1). \square

Proposition 3.2. *Let (M, g) be a closed Riemannian spin surface. Moreover, assume that $\psi \in \Gamma(\Sigma^+ M)$ is a harmonic spinor, that is $D\psi = 0$. Then the following formula holds*

$$N_0 = -\frac{\chi(M)}{2}, \quad (3.2)$$

where N_0 is the sum of the order of the zeros of ψ and $\chi(M)$ denotes the Euler characteristic of the surface.

Proof. By assumption $\psi \in \Gamma(\Sigma^+ M)$, hence we may write $\psi = f\psi^+$, where $f: M \rightarrow \mathbb{C}$ is a complex-valued function and ψ^+ a parallel spinor. By a direct calculation it follows that the condition $D\psi = 0$ is equivalent to the fact that f is holomorphic. Hence, we already know that ψ has isolated zeros, which would also follow from [6]. Choosing Riemannian normal coordinates the metric takes the form $g(\partial_{\bar{z}}, \partial_{\bar{z}}) = g(\partial_z, \partial_z) = 0$, $g(\partial_z, \partial_{\bar{z}}) = g(\partial_{\bar{z}}, \partial_z) = 1$. We calculate

$$d|f|^2 = f \nabla \bar{f} + \bar{f} \nabla f = \bar{f} \frac{\partial f}{\partial z} \frac{\partial}{\partial \bar{z}} + f \frac{\partial \bar{f}}{\partial \bar{z}} \frac{\partial}{\partial z},$$

where we used that f is holomorphic. Thus, we find

$$|d|f|^2|^2 = 2|f|^2 |\nabla f|^2 \quad (3.3)$$

yielding

$$|d|\psi|^2|^2 = 2|\psi|^2|\nabla\psi|^2.$$

Inserting this into (3.1) and using $D\psi = 0$ we obtain

$$\Delta \log |\psi| = \frac{K}{2}.$$

Integrating over the surface M and applying (2.3) then proves the assertion. \square

Remark 3.3. The same formula holds true if $\psi \in \Gamma(\Sigma^- M)$ solves $D\psi = 0$.

The last Proposition is a special case of a structure Theorem for Dirac-harmonic maps between surfaces, which was proven in [22], Theorem 4.2. In addition, the statement was also proven using the Poincaré-Hopf Index Theorem in [16], Theorem 4.12.

Now, we turn to the analysis of *twistor spinors*, for more details on them see the book [9].

Definition 3.4. Let (M, g) be a n -dimensional Riemannian spin manifold. A twistor spinor is a section ψ of ΣM satisfying

$$P\psi = 0, \tag{3.4}$$

where $P_X\psi := \nabla_X\psi + \frac{1}{n}X \cdot D\psi$ for every $X \in TM$.

For twistor spinors we will prove the following

Proposition 3.5. *Let (M, g) be a closed Riemannian spin surface and let $\psi \in \Gamma(\Sigma^+ M)$ be a twistor spinor. Then we have*

$$N = \chi(M), \tag{3.5}$$

where N is the sum of the order of the zeros of ψ . The same formula also holds for $\psi \in \Gamma(\Sigma^- M)$.

Proof. If ψ is a twistor spinor on a n -dimensional Riemannian manifold, then it satisfies

$$D^2\psi = R\frac{n}{4(n-1)}\psi,$$

see for example [9], p.24. Similar to the case of harmonic spinors it can be checked that being a twistor spinor on a surface is equivalent to the fact that $f: M \rightarrow \mathbb{C}$ is holomorphic for $\psi := f\psi^+$, where ψ^+ is a parallel spinor. Using $|\psi|^2|\nabla\psi|^2 = 2|d|\psi|^2|^2$ as in the case of harmonic spinors (3.3) and the Bochner formula (3.1) we find

$$\Delta \log |\psi| = -\frac{1}{2}R.$$

Integrating this equation using (2.3) and the Gauß-Bonnet theorem completes the proof. \square

Corollary 3.6. *This confirms the well-known fact that the only closed surfaces admitting non-trivial twistor spinors are S^2 and T^2 .*

As a next step we turn to the analysis of *Killing spinors*, which are twistor spinors that are also eigenspinors of the Dirac operator.

Definition 3.7. For a complex number α , an α -Killing spinor on (M, g) is a section ψ of ΣM satisfying

$$\nabla_X\psi = \alpha X \cdot \psi \tag{3.6}$$

for every $X \in TM$.

Using our framework we can also prove the following well-known (see for example Proposition A.4.1 in [14])

Proposition 3.8. *Let (M, g) be a closed Riemannian spin surface and let $\psi \in \Gamma(\Sigma^+ M)$ be a Killing spinor. Then the nodal set of ψ is empty.*

Proof. It is well-known that a Killing spinor satisfies

$$D\psi = -2\alpha\psi, \quad R = 8\alpha^2.$$

From (3.1) we obtain

$$\Delta \log |\psi| = -\frac{R}{2} + \lambda^2 = -4\alpha^2 + 4\alpha^2 = 0.$$

The result follows by integration over M and applying (2.3). \square

The spinorial Weierstraß representation for CMC-surfaces in \mathbb{R}^3 is governed by the equation

$$D\psi = \mu|\psi|^2\psi, \quad \|\psi\|_{L^4} = 1, \quad (3.7)$$

where μ denotes some real constant. The condition on the L^4 norm of ψ normalizes the volume of the surface to be equal to 1. We refer to [1] and references therein for more details. We can give an estimate on the nodal set of solutions of (3.7), which was already proven in [1].

Proposition 3.9. *The zero set of solutions of (3.7) can be estimated as*

$$N \leq \frac{\mu^2}{4\pi} - \frac{\chi(M)}{2}. \quad (3.8)$$

Proof. Again, by the main result of [6] we know that the nodal set of solutions of (3.7) is discrete. We compute

$$\langle \psi, D^2\psi \rangle = \langle \psi, D(\mu|\psi|^2\psi) \rangle = \mu \underbrace{\langle \psi, (\nabla|\psi|^2) \cdot \psi \rangle}_{=0} + \mu \langle \psi, |\psi|^2 D\psi \rangle = \mu^2 |\psi|^6,$$

where we used the skew-symmetry of the Clifford multiplication and (3.7) in the last step. Using that $|T|^2 \geq 2\mu^2|\psi|^4$ in this case we obtain from (1.5)

$$\mu^2 |\psi|^4 \geq K - 2\Delta \log |\psi|$$

and integrating over the surface M using (2.3) yields the assertion. \square

4. APPLICATIONS

In this section we give some applications of Theorem 1.3.

Example 4.1. We can use (1.7) to estimate the nodal set of the eigenspinor belonging to the first eigenvalue on S^2 with the round metric. The Dirac-spectrum on $M = S^n$ for $n \geq 2$ equipped with the round metric is well-known (see for example [4]): The eigenvalues of the Dirac operator are

$$\lambda_k = \pm\left(\frac{n}{2} + k\right), \quad k \in \mathbb{N}.$$

Thus, from (1.7) we find that the zero set of an eigenspinor belonging to the first eigenvalue on S^2 is empty. This result is also well-known: On the sphere with the round metric the eigenspinors belonging to the eigenvalues $\pm\frac{n}{2}$ are Killing spinors with $\alpha = \pm\frac{n}{2}$, see [14], Example A.1.3 and references therein. Thus, their nodal set is empty. This statement can also be obtained by an explicit calculation: In [15], Chapter 4, it is shown that on the sphere S^n with the round metric the eigenspinors corresponding to the eigenvalues $\lambda = \pm\frac{n}{2}$ are nowhere zero.

We can generalize this statement to the case of an arbitrary metric with positive curvature. Recall the following upper bound for the first eigenvalue from [8], Corollary 1:

Lemma 4.2. *Let (M^{2m}, g) be a compact even-dimensional spin manifold of positive sectional curvature $0 \leq K$ and let K_{\max} be the maximum of K . Then the first eigenvalue of the Dirac operator D is bounded by*

$$\lambda_1^2 \leq 2^{\frac{n}{2}-1} \frac{n}{2} \max_M K^M. \quad (4.1)$$

Lemma 4.3. *Assume that $M = S^2$ with a metric of positive curvature. Then we can estimate the nodal set of the eigenspinor belonging to the first eigenvalue on S^2 as*

$$N_1 \leq \frac{\text{Vol}(S^2, g)}{4\pi} \max_{S^2} K - 1. \quad (4.2)$$

Proof. Applying (4.1) on S^2

$$\lambda_1^2 \leq \max_{S^2} K$$

and using (1.7) proves the result. \square

We can also give an upper bound for the nodal set on hyperbolic surfaces. In this case we will make use of an inequality given by Lott [20]:

Theorem 4.4. *Let (M^n, g) be an n -dimensional closed spin manifold with $n \geq 2$. Then for any conformal class $[g]$ on M^n , there exists $b[g] > 0$ such that*

$$\lambda_1^2 \leq b[g] \sup_M (-K) \quad (4.3)$$

for any $g \in [g]$ with $R_g < 0$.

Proposition 4.5. *Let M be a closed surface with $\chi(M) < 0$. Then the nodal set of the eigenspinor corresponding to the first eigenvalue of D can be estimated as*

$$N_1 \leq \frac{\text{Vol}(M, g)b[g]}{4\pi} \sup_M (-K) - \frac{\chi(M)}{2}. \quad (4.4)$$

Proof. This follows directly from (1.5) and (4.3). \square

The Willmore energy of a surface $M \subset \mathbb{R}^3$ is defined by

$$W(M) := \int_M H^2 d\mu,$$

where H denotes the mean curvature of M in \mathbb{R}^3 . It was shown in [5] that the Willmore energy $W(M)$ can be estimated with the help of the spectrum of the Dirac operator via

$$W(M) \geq \lambda^2 \text{Vol}(M, g).$$

Using (1.5) this directly implies

Proposition 4.6. *We can estimate the Willmore energy as*

$$W(M) \geq 2\pi\chi(M) + 4\pi N_k. \quad (4.5)$$

Using the Weyl-asymptotic for linear elliptic operators we can give an estimate on the nodal set of eigenspinors for large eigenvalues.

Proposition 4.7. *Let M be a closed surface and let $\psi_k \in \Gamma(\Sigma M)$ be an eigenspinor of D with eigenvalue λ_k . For large values of k we find the following estimate*

$$N_k \leq \frac{4\pi^3}{\text{Vol}(M, g)} k^2 - \frac{\chi(M)}{2}. \quad (4.6)$$

Proof. Recall the Weyl asymptotic for large eigenvalues of a linear elliptic differential operator

$$(\lambda_k)^{\frac{n}{2}} \sim \frac{(2\pi)^n k}{\text{Vol}(M, g)\omega_n}, \quad \omega_n = \frac{n^{n/2}}{\Gamma(\frac{n}{2} + 1)}.$$

The result follows by combining the Weyl asymptotic with (1.7). \square

REFERENCES

- [1] Bernd Ammann. The smallest Dirac eigenvalue in a spin-conformal class and cmc immersions. *Comm. Anal. Geom.*, 17(3):429–479, 2009.
- [2] Bernd Ammann and Christian Bär. Dirac eigenvalue estimates on surfaces. *Math. Z.*, 240(2):423–449, 2002.
- [3] Christian Bär. Lower eigenvalue estimates for Dirac operators. *Math. Ann.*, 293(1):39–46, 1992.
- [4] Christian Bär. The Dirac operator on space forms of positive curvature. *J. Math. Soc. Japan*, 48(1):69–83, 1996.
- [5] Christian Bär. Extrinsic bounds for eigenvalues of the Dirac operator. *Ann. Global Anal. Geom.*, 16(6):573–596, 1998.
- [6] Christian Bär. Zero sets of solutions to semilinear elliptic systems of first order. *Invent. Math.*, 138(1):183–202, 1999.

- [7] Christian Bär and Paul Schmutz. Harmonic spinors on Riemann surfaces. *Ann. Global Anal. Geom.*, 10(3):263–273, 1992.
- [8] Helga Baum. An upper bound for the first eigenvalue of the Dirac operator on compact spin manifolds. *Math. Z.*, 206(3):409–422, 1991.
- [9] Helga Baum, Thomas Friedrich, Ralf Grunewald, and Ines Kath. *Twistors and Killing spinors on Riemannian manifolds*, volume 124 of *Teubner-Texte zur Mathematik [Teubner Texts in Mathematics]*. B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1991. With German, French and Russian summaries.
- [10] Th. Friedrich. Der erste Eigenwert des Dirac-Operators einer kompakten, Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung. *Math. Nachr.*, 97:117–146, 1980.
- [11] Th. Friedrich. Zur Abhängigkeit des Dirac-operators von der Spin-Struktur. *Colloq. Math.*, 48(1):57–62, 1984.
- [12] Thomas Friedrich. *Dirac-Operatoren in der Riemannschen Geometrie*. Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 1997. Mit einem Ausblick auf die Seiberg-Witten-Theorie. [With an outlook on Seiberg-Witten theory].
- [13] Thomas Friedrich. Solutions of the Einstein-Dirac equation on Riemannian 3-manifolds with constant scalar curvature. *J. Geom. Phys.*, 36(3-4):199–210, 2000.
- [14] Nicolas Ginoux. *The Dirac spectrum*, volume 1976 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.
- [15] Andreas Hermann. Dirac eigenspinors for generic metrics. *PhD-thesis*, 2012.
- [16] Andreas Hermann. Zero sets of eigenspinors for generic metrics. *Comm. Anal. Geom.*, 22(2):177–218, 2014.
- [17] Nigel Hitchin. Harmonic spinors. *Advances in Math.*, 14:1–55, 1974.
- [18] Eui Chul Kim and Thomas Friedrich. The Einstein-Dirac equation on Riemannian spin manifolds. *J. Geom. Phys.*, 33(1-2):128–172, 2000.
- [19] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.
- [20] John Lott. Eigenvalue bounds for the Dirac operator. *Pacific J. Math.*, 125(1):117–126, 1986.
- [21] R. Schoen and S. T. Yau. *Lectures on harmonic maps*. Conference Proceedings and Lecture Notes in Geometry and Topology, II. International Press, Cambridge, MA, 1997.
- [22] Ling Yang. A structure theorem of Dirac-harmonic maps between spheres. *Calc. Var. Partial Differential Equations*, 35(4):409–420, 2009.

TU WIEN, INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE, WIEDNER HAUPTSTRASSE 8–10, A-1040 WIEN

E-mail address: volker@geometrie.tuwien.ac.at